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A model for the porosity dependence of Young's modulus in brittle solids based on crack opening displacement

V. D. KRSTIC, W. H. ERICKSON
Physical Metallurgy Research Laboratories CANMET, Energy, Mines and Resources Canada, Ottawa, Canada

A crack opening displacement concept has been introduced to model the porosity dependence of Young's modulus in polycrystalline and single phase solids. In developing the theoretical model, it is assumed that each cylindrical cavity possesses radial cracks and spherical pores possess annular flaws. When an external stress is applied on such a solid, its elastic response is shown to be governed by the pore size, the width of an annular flaw, the number of pores (or pore volume fraction) and the flaw to pore size ratio. The validity of the present approach is tested against a number of experimental data.

1. Introduction

It is well known that the presence of pores in brittle solids strongly affects their elastic, mechanical and other properties. So far, several theories have been proposed to relate the elastic modulus of porous materials to the number of pores and their volume fraction. The first attempts to characterize the elastic response of a solid containing pores were based on a semi-empirical equation of the form [1, 2, 3] $E = E_0 \exp(-bV)$, where E_0 is the Young's modulus of a pore-free solid, V is the pore volume fraction and b is an empirical constant. This exponential equation was widely used to fit experimental data on porosity dependence of Young's modulus.

Theoretical expressions for the effect of porosity on Young's modulus have been derived by several investigators. Hill [4] and Budiansky [5] independently suggested a self-consistent method to describe the change of elastic constants with porosity.

One approach to characterize the elastic behaviour of a solid containing porosity is to assume pores to take the shape of nearly flat oblate spheroids, and treat them as a crack-like defect. Based on this argument, numerous theoretical models have been presented which take into account the effect of crack size and its density [6–9]. MacKenzie [9] was the first to recognize the importance of stress concentration in determining the elastic behaviour of solids under stress.

More recent theories assume that the only disturbances that affect the elastic properties of a porous solid are pores of various shapes [10–14]. When interpreting experimental data on the porosity dependence of Young's modulus it is worth pointing out that, when expressed in terms of elastic strain on a stress–strain curve, the introduction of pores leads simultaneously to the reduction of Young's modulus and to an equivalent increase of elastic strain for a given stress. One possible explanation for the increase of elastic strain is due to crack opening displacement caused by the presence of radial and/or annular cracks associated with pores. It is this concept on which the present theoretical model is based.

When dealing with porous solids, it is important to recognize the significance of both pores and cracks associated with pores. Such a crack–pore configuration was found to be the dominant failure precursor in a number of porous solids [15, 16]. It has been demonstrated experimentally [15] and theoretically [16] that the stress concentration due to the presence of pores and the crack–pore stress field interaction effects are so large that they cannot be neglected. It is believed that these stresses also play an important role in determining the overall elastic response of a solid containing pores. The theoretical analysis presented in this paper is therefore based on the crack opening displacement and the stress concentration, where it is assumed that the additional elastic strain of a porous solid under stress comes from the opening of all cracks associated with pores. Based on this, an expression for the porosity dependence of Young's modulus is developed and compared with experimental data.

2. Development of the model

2.1. Cylindrical cavity

Consider a solid containing randomly distributed, noninteracting, cylindrical cavities with two radial cracks, Fig. 1. The total displacement at remote points is

$$\Delta_{\text{tot}} = \Delta_c + \Delta_{\text{nc}} \tag{1}$$

where Δ_c is the displacement due to the presence of cracks and Δ_{nc} is the displacement in the absence of cracks. The displacement due to the presence of cracks is [17]

$$\Delta_c = \frac{4\sigma C}{E'} V_1 \tag{2}$$

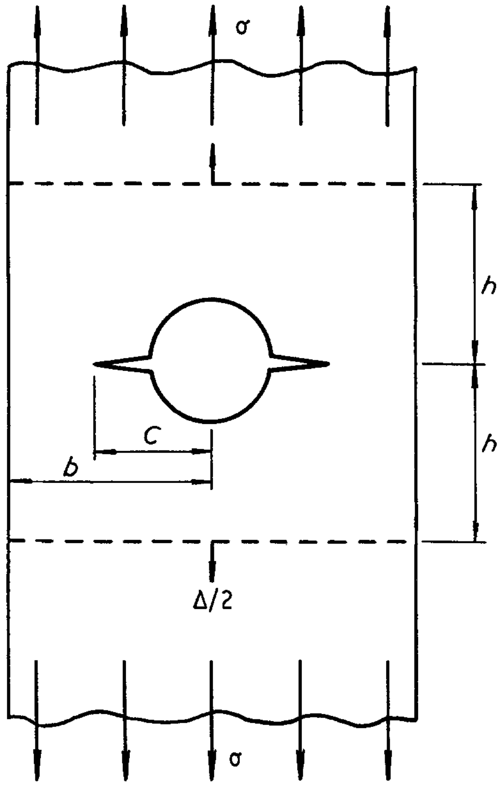

Figure 1 Displacement at remote points of a solid containing cylindrical cavity under applied stress.

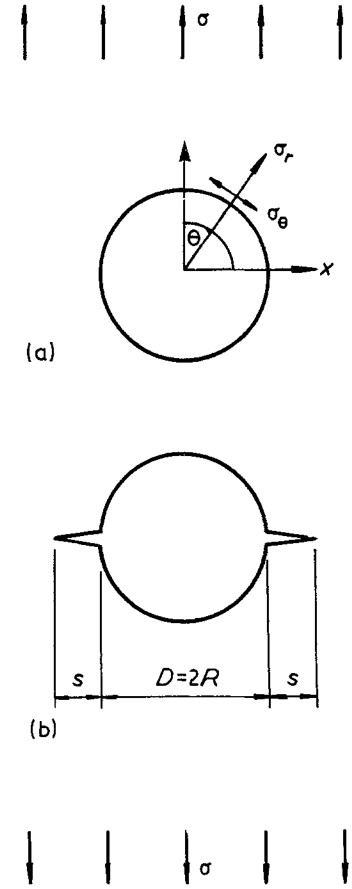

Figure 2 (a) Stress concentration distribution around cavity in the absence of crack: (b) Annular or radial cracks emanating from the cavity surface.

where σ is the applied stress, $C(=R + s)$ is the total crack length, R is the pore radius, s is the radial flaw size,

$$\begin{aligned} V_1 = &-1.071 + 0.250(C/b) - 0.357(C/b)^2 \\ &+ 0.121(C/b)^3 - 0.047(C/b)^4 + 0.008(C/b)^5 \\ &- 1.071(C/b)^{-1} \ln(1 - C/b) \end{aligned} \quad (3)$$

$E' = E$ (plane stress), $E' = E/(1 - \nu^2)$ (plane strain) and ν is Poisson's ratio. For the limiting case where $C/b \to 0$, and neglecting higher order terms beyond (C/b), Equation 2 reduces to the simple form

$$\Delta_c = \frac{3.14\sigma C^2}{bE'} \quad (4)$$

The displacement of a solid containing no cracks is:

$$\Delta_{nc} = \frac{\sigma}{E} 2h \quad (5)$$

where h is the sample half length. The total strain of a solid containing $N_s(=N/2h2b)$ cracks per unit surface is

$$\varepsilon_c = \frac{\Delta_{tot}}{2h} = \frac{\sigma}{E_0}[1 + 6.28N_s(1 - \nu^2)C^2] \quad (6)$$

where E_0 is the Young's modulus of a cavity-free solid, ν_0 is Poisson's ratio and N is the total number of cracks. Hence, the total strain of a solid containing cylindrical cavities with radial cracks is:

$$\varepsilon_c = \varepsilon_0 + 6.28\sigma N_s(1 - \nu^2)C^2 \quad (7)$$

where ε_0 is the strain of a crack-free solid.

In developing Equation 7 it was assumed that the opening of all cracks is solely due to a remotely applied stress. However, for a cavity-crack configuration depicted in Fig. 2, the stress concentration is developed due to the presence of a cavity, which is tensile at $\theta = \pi/2$ (see Fig. 2) and is represented by the expression [18]

$$\sigma_\theta = \sigma[1/2\,(R/x)^2 + 3/2\,(R/x)^4 + 1] \quad (8)$$

where x is the distance from the centre of the cavity and R is the cavity radius. The effect of stress concentration due to the presence of cylindrical cavities on the total strain is obtained by substituting σ in Equation 7 by σ_θ from Equation 8

$$\begin{aligned} \varepsilon_c = &\varepsilon_0 + 6.28N_s\sigma(1 - \nu^2)R^2[1/2 + 3/2(1 + s/R)^2 \\ &+ (1 + s/R)^2] \end{aligned} \quad (9)$$

Hence, it follows from Equation 9 that the effective Young's modulus is

$$\begin{aligned} E = &E_0\{1 + 6.28N_s(1 - \nu^2)R^2[1/2 + 3/2(1 + s/R)^2 \\ &+ (1 + s/R)^2]\}^{-1} \end{aligned} \quad (10)$$

For the limiting case of $s \to 0$, Equation 10 reduces to the simple form

$$E = E_0[1 + 18.84(1 - \nu^2)N_sR^2]^{-1} \quad (11)$$

An alternative approach to arriving at the expression relating the Young's modulus to the crack density is the complience method. This procedure, shown in the Appendix, indicates that both crack opening and complience procedures are equally valid and lead to the same result (See Equations 4 and A6).

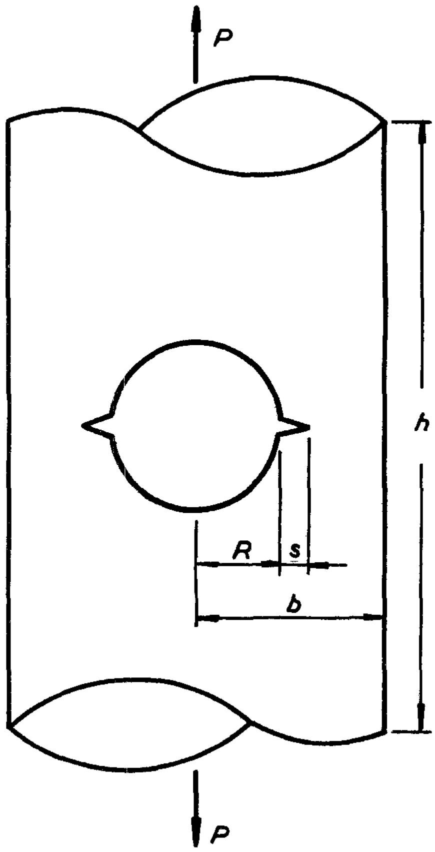

Figure 3 Round specimen with centrally located spherical pore subjected to external load, P.

2.2. Spherical cavity

Another case of interest is the three-dimensional spherical pore subjected to an external stress at infinity. The tensile stress distributed along the equator of a spherical cavity is [18] (for $\theta = \pi/2$):

$$\sigma_\theta = \sigma\{[(4 - 5\nu)/2(7 - 5\nu)](R/x)^3 + [9/2(7 - 5\nu)](R/x)^5 + 1\} \quad (12)$$

The stress intensity factor (K_1) for a penny-shaped crack of length, C, located in the centre of a round specimen (Fig. 3) is [17]

$$K_1 = \sigma_{\text{net}}(\pi C)^{1/2} V_3 \quad (13)$$

where

$$\sigma_{\text{net}} = \frac{P}{\pi(b^2 - C^2)}, \quad \sigma = \frac{P}{\pi b^2}, \quad A = \pi C^2 \quad (14)$$

and

$$V_3 = \frac{2}{\pi}[1 + 1/2(C/b) - 5/8(C/b)^2 + 0.421(C/b)^3] \quad (15)$$

Neglecting the higher order terms beyond (C/b) for $C/b \to 0$, the complience of a solid containing a single penny-shaped crack may be obtained by combining Equations 13, A1 and A2 to yield

$$\mathrm{d}Q = \frac{16}{bE'\pi^2}\ell^2\mathrm{d}\ell \quad (16)$$

where $\ell = C/b$. Integration of Equation 16 yields

$$Q = 16/3\pi^2 bE'\,(C/b)^3 \quad (17)$$

The displacement of a solid at remote points due to the presence of a single isolated crack is

$$\Delta_c = QP = 16\sigma C^3/3bE'\pi^2 \quad (18)$$

Now, the total strain due to N^p cracks per unit volume is (from Equations 1 and 5 and Fig. 3)

$$\varepsilon_p = \Delta_t/h = \sigma[1/E_0 + 16(1 - \nu^2)N^p C^3/3E_0] \quad (19)$$

which leads to the equation relating the total strain to the crack density and crack length

$$\varepsilon_p = \varepsilon_0 + 16\sigma(1 - \nu^2)N^p C^3/3E_0 \quad (20)$$

From Equation 20, it can easily be shown that the effective Young's modulus is

$$E = E_0[1 + 16(1 - \nu^2)N^p C^3/3]^{-1} \quad (21)$$

which recovers the result obtained by Hasselman and Singh [23] employing the energy balance criteria.

The effect of stress concentration on elastic strain of a solid containing spherical pores can now be obtained by replacing σ in Equation 20 by σ_θ from Equation 12 to yield

$$\varepsilon_p = \varepsilon_0 + \frac{16\sigma(1 - \nu^2)N^p(R + s)^3}{3E_0} \times \left[\frac{4 - 5\nu}{2(7 - 5\nu)}\frac{R^3}{(R + s)^3} + \frac{9}{2(7 - 5\nu)}\frac{R^5}{(R + s)^5} + 1\right] \quad (22)$$

This immediately leads to an expression for the effective Young's modulus

$$E = E_0\left\{1 + \frac{16N^p(1 - \nu^2)R^3}{3}\left[(1 + s/R)^3 + \frac{9}{2(7 - 5\nu)(1 + s/R)^2} + \frac{4 - 5\nu}{2(7 - 5\nu)}\right]\right\}^{-1} \quad (23)$$

Assuming that the stress field interaction of neighbouring pores and cracks is negligible, it is of interest now to develop the relationship between the Young's modulus and the pore volume fraction. The relationship between the number of pores per unit volume, N^p, and their volume fraction (9 9)is given by the expression

$$N^p = \frac{3V}{4\pi R^3} \quad (24)$$

Combining Equations 23 and 24 results in

$$E = E_0\left\{1 + \frac{12V(1 - \nu^2)}{\pi}\left[(1 + s/R)^3 + \frac{9}{2(7 - 5\nu)(1 + s/R)^2} + \frac{4 - 5\nu}{2(7 - 5\nu)}\right]\right\}^{-1} \quad (25)$$

For a solid possessing very small angular flaws

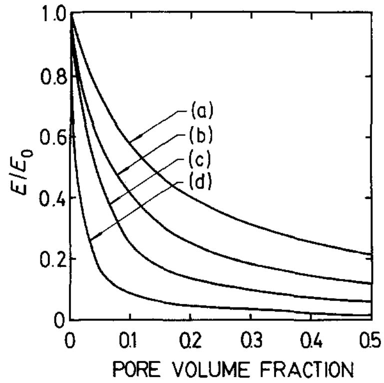

Figure 4 Predicted change of Young's modulus (from Equation 25) with porosity for given s/R values and $\nu = 0.2$. s/R = (a) 0.001, (b) 0.5, (c) 1, (d) 2.

($s \rightarrow 0$), Equation 25 simplifies to (for $\nu = 0.2$)

$$E = E_0 [1 + 24V(1 - \nu^2)/\pi]^{-1} \qquad (26)$$

Equation 26 correctly predicts the extreme case of $V \rightarrow 0$ where the Young's modulus approaches that of a pore-free solid. For the other extreme where $V \rightarrow 1$, the Young's modulus should be equal to zero. However, Equation 25 shows that at $V \rightarrow 1$ the Young's modulus approaches a small but finite value instead of zero. To satisfy this later boundary condition Equation 25 may be modified to read

$$\frac{E}{E_0} = (1 - V)\left\{1 + \frac{12V(1 - \nu^2)}{\pi}\left[(1 + s/R)^3 + \frac{9}{2(7 - 5\nu)(1 + s/R)^2} + \frac{4 - 5\nu}{2(7 - 5\nu)}\right]\right\}^{-1} \qquad (27)$$

Fig. 4 illustrates the change of Young's modulus with pore volume fraction and s/R. Although Fig. 4 and Equation 27 show a strong s/R dependence of Young's modulus at all levels of porosity, the most significant drop of Young's modulus appears to occur at low and intermediate levels of porosity (< 30 vol %).

When comparing Equations 10 and 27, it is clear that both equations exhibit the same fundamental change of Young's modulus with volume fraction of cavities. Nevertheless, Equation 10 shows somewhat

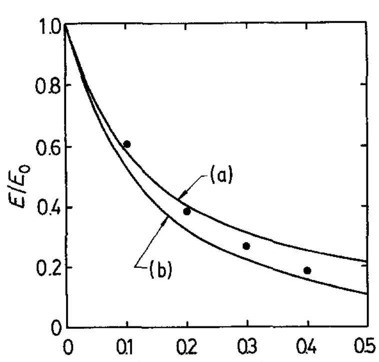

Figure 5 Comparison between predicted and measured values for Young's modulus in a polycrystalline alumina. The data are collected from eight different sources [3]. E_0 was obtained from linear extrapolation at $V = 0$. (●) Experimental result, (a) Equation 25, $s/R = 0.001$, (b) Equation 27, $s/R = 0.001$.

greater sensitivity of elastic modulus to the presence of cylindrical cavities.

3. Discussion

As shown by the present analysis, the elastic response of a solid containing pores is governed by the number of pores or their volume fraction, the pore size and the s/R ratio. Because of the very fast decay of the stress concentration with distance from the surface of the cavity, its effect on elastic modulus is most significant at small values of s/R. At large s/R ratios ($s/R >$ 1 to 2), such as may be the case with highly cracked solids, the contribution of localized stresses becomes negligible and the porous solid exhibits the same general behaviour as if it contained no cavities. This means that, for a given radial/annular flaw, a solid that contains a large number of small cavities will have its Young's modulus less affected by the presence of cavities compared to a solid with a small number of large cavities. It can be inferred from the present analysis that the most advantageous system in preserving a high value for its Young's modulus, is the one that contains no radial or annular cracks.

For the comparison of the theory with experiments, Fig. 5 displays the experimental data for the porosity dependence of Young's modulus in a polycrystalline alumina together with calculated values. In calculating the theoretical values for elastic modulus, a constant s/R was assumed. Although experimental data on the relationship between the pore size and the radial (annular) flaw size are limited [15], there is no evidence that the length of a radial crack is limited by, or related to the pore size.

It is more realistic to expect some form of relationship between the radial (annular) flaw size, and the grain size. In spite of the fact that in most polycrystalline brittle solids a unique, well defined relationship between the flaw size and the grain size does not exist [22], a much stronger effect of grain size on elastic modulus is anticipated. This may particularily be significant with anisotropic polycrystalline solids where extensive spontaneous cracking occurs when the grain size exceeds a certain critical value [22]. Very weak or no connection between the grain size and the flaw size is expected to exist in thermally isotropic systems where the grain size and the flaw size are independent parameters [22]. In systems where the equality between the grain and the flaw size exists, the contribution of grain size can be included into the equation for the porosity dependence of Young's modulus by writing $s = \Gamma$ where Γ is the grain size.

To further test the validity of the present approach, Fig. 6 compares the computed and measured values for Young's modulus as a function of pore volume fraction in a polycrystalline spinel [21]. Close examination of Fig. 6 indicates that, for low porosity (< 20%) the theory slightly underestimates the measured Young's modulus, whereas at intermediate porosity levels (20–40%), Equation 25 appears to fit experimental data reasonably well. Another note of interest is the much better agreement between the theory and experiments at low s/R, indicating that large radial/annular cracks are not anticipated in a porous polycrystalline

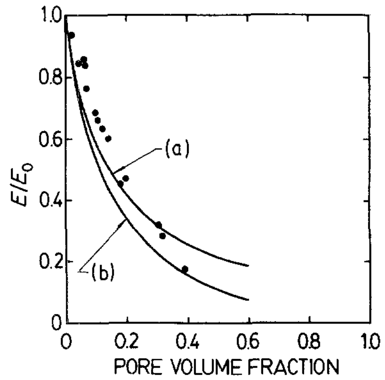

Figure 6 Computed and measured variation of Young's modulus as a function of porosity in a polycrystalline spinel [21]. E_0 was obtained from linear extrapolation at $V = 0$. (●) Experimental results (a) Equation 25, $s/R = 0.001$, (b) Equation 27, $s/R = 0.001$.

spinel. This certainly does not apply to highly anisotropic polycrystalline ceramics where extensive spontaneous cracking frequently occurs even prior to load application.

So far, the discussion has centred on the effect of spherical voids on elastic modulus. Due to the lack of experimental data on the effect of cylindrical cavities on elastic modulus, the direct correlation between predicted and measured values is not possible. However, it is immediately evident from Equations 10 and 23 that, for a given crack density, somewhat more severe degradation of Young's modulus is expected in a solid containing cylindrical cavities. Furthermore, this finding also serves to suggest that the cavity shape has a certain role in determining the elastic response of a porous solid as discussed elsewhere [12]. However, the complexity of the stress distribution around a cavity of arbitrary shape (other than spherical or cylindrical) precludes rigorous evaluation of the problem.

Although, the present theoretical analysis is valid for any s/R value, for the limiting case $s/R \to 0$, the opening of crack-like defects such as pores with annular flaws, and the elastic response of a solid containing cavities is expected to be governed by the condition of crack initiation rather than crack extension. This mechanism may be of certain significance in glasses where pores appear to exhibit rather smooth surfaces with very small or no radial cracks present [15], at least prior to stress application. In polycrystalline ceramics, on the other hand, radial flaws of various length appear to be always associated with pores regardless of other microstructural features, forcing pores to behave like sharp cracks [15].

4. Conclusions

A crack opening is found to be a useful approach in developing the relationship between the Young's modulus and the volume fraction porosity. It is found that when a porous solid is subjected to uniform external stress, its displacement, and consequently its elastic strain at remote points (from the crack), is governed by the opening of all radial or annular cracks associated with cavities. This crack opening is further increased by the action of stress concentration induced by the presence of cavities.

Due to different distributions of stresses around spherical and cylindrical cavities, the Young's modulus is found to be more sensitive to the presence of cylindrical cavities.

Appendix

The relationship between the strain energy release rate, G, the load, P, and complience, Q, is given by the well known expression

$$G = \frac{P^2}{2}\frac{\partial Q}{\partial A} \tag{A1}$$

where A is the crack surface area.

From the linear elastic fracture mechanics, the strain energy release rate is related to the stress intensity factor via the equation

$$G = \frac{K^2}{E'} \tag{A2}$$

The stress intensity factor of a crack as shown in Fig. 1 [17] is

$$K = \sigma(\pi C)^{1/2} V_2 \tag{A3}$$

where $V_2 = 1 + 0.128(C/b) - 0.288(C/b)^2 + 1.525(C/b)^3$. For the limiting case $C/b \to 0$ and neglecting the higher order terms beyond (C/b), Equation A3 assumes the form

$$K = \sigma(\pi C)^{1/2}[1 + 0.128(C/b)] \tag{A4}$$

Also, from Fig. 1, $P = \sigma(B2b)$ and $A = B2C$. From Equations A1 and A2

$$\mathrm{d}Q = \frac{2}{P^2}\frac{K^2}{E'}\,\mathrm{d}A \quad \text{and} \quad Q = 3.14\sigma C^2/2Bb^2E' \tag{A5}$$

where B is the sample thickness. Finally, we arrive at the expression for the displacement of a solid due to the presence of cracks

$$\Delta_c = QP = 3.14\sigma C^2/bE' \tag{A6}$$

Equation A6 is identical with Equation 4 which justifies the use of complience method to relate the displacement at remote points with the crack opening.

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